2. Radiation and Retarded Potentials

2.3 Radiation from a charged particle with acceleration parallel to velocity

- Prior to studying the radiation produced by a moving point charge we present the **General theory of radiation**
- As we know, radiation is induced by the acceleration component of the E-field

\[
\vec{E}_a = \frac{q}{4\pi\varepsilon_0 c} \left[ \hat{R} \times \left( (\hat{R} - \hat{\beta}) \times \hat{\beta} \right) \right]_{\text{ret}} \left( 1 - \hat{\beta} \cdot \hat{R} \right) \hat{R} \]

and the corresponding acceleration component of the B-field, \( \vec{B}_a = \frac{1}{c} \left[ \hat{R} \right]_{\text{ret}} \times \vec{E}_a \)

- Note, both \( \vec{E}_a \) and \( \vec{B}_a \) are both perpendicular to the unit vector \( \hat{R} \)
- The radiation power per unit area is given by the Poynting vector:

\[
\vec{P}_{\text{rad}} = \frac{1}{\mu_0} \left( \vec{E}_a \times \vec{B}_a \right) = \frac{1}{\mu_0 c} \vec{E}_a \times \left[ \left[ \hat{R} \right]_{\text{ret}} \times \vec{E}_a \right]
\]

\[
\Rightarrow \vec{P}_{\text{rad}} = \frac{1}{\mu_0 c} E^2_{a} \left[ \hat{R} \right]_{\text{ret}}
\]

\( \left[ \hat{P}_{\hat{R}} \right]_{\text{ret}} \) = power radiated per unit area at a distance \( [R]_{\text{ret}} \) at present time \( t = \frac{d^2W}{dt} \frac{1}{R^2(t')d\Omega} \)

- The energy radiated (lost) by the particle during a finite period of time from \( t_1 \) to \( t_2 \) is:

\[
\frac{dW}{d\Omega} = \int_{t_1 - R(t')/c}^{t_2 + R(t')/c} dt \left[ \hat{P}_{\hat{R}} R^2 \right]_{\text{ret}} = \int_{t_1}^{t_2} dt \left[ \hat{P}_{\hat{R}} R^2 \right]_{\text{ret}} \frac{dt}{dt'}
\]

Now, \( t = t' + \frac{R(t')}{c} \Rightarrow \frac{dt}{dt'} = 1 + \frac{1}{c} \frac{dR}{dt'} \)

\[
\frac{1}{c} \frac{dR}{dt'} = \left[ \nabla, \vec{R} \hat{\beta} \right]_{\text{ret}} = \left[ \hat{R}, \hat{\beta} \right]_{\text{ret}} = -\left[ \hat{R}, \hat{\beta} \right]_{\text{ret}} \text{ (as we saw earlier)}
\]
• Thus, the power radiated by an accelerated charged particle:

\[
\frac{d^2W}{dt'd\Omega} = \frac{dP}{d\Omega} = \left(\vec{P} \cdot \hat{R}\right)(1-\beta \hat{R})R^2
\]

\[
= \frac{q^2}{16\pi^2\varepsilon_0c} \left[\left(\hat{R} \times \left(\hat{R} \times \hat{\beta}\right)\right)^2\right]_{net}
\]

• Note, although \(\vec{P}\) is derived in co-moving frame of the charge \(q\), it is valid for any other frame as well, including the laboratory frame. This means that \(\vec{P}\) is invariant under Lorentz transformations. More will follow on this later in the final section on relativity.

We now consider the case of acceleration parallel to velocity or \(\vec{\beta} \parallel \hat{\beta}\) and \(\vec{\beta} \times \hat{\beta} = 0\):

\[
\frac{d^2W}{dt'd\Omega} = \frac{dP}{d\Omega} = \frac{q^2}{16\pi^2\varepsilon_0c} \left[\left(\hat{R} \times \hat{\beta}\right)^2\right]_{net}
\]

Now \(\hat{A} \times \hat{B} \times \hat{C} = (\hat{A} \cdot \hat{C})\hat{B} - (\hat{A} \cdot \hat{B})\hat{C} \Rightarrow \hat{R} \times \hat{R} = \hat{R} = \beta \cos \theta \hat{R} - \hat{\beta}\)

and \(\left(\hat{R} \times \hat{R} \times \hat{\beta}\right)^2 = \left(\beta \cos \theta\right)^2 + \beta^2 - 2\left(\beta \cos \theta\right)^2 = \beta^2 \sin^2 \theta\)

\[
\frac{dP}{d\Omega} = \frac{q^2}{16\pi^2\varepsilon_0c} \left[\frac{\beta^2 \sin^2 \theta}{(1-\beta \cos \theta)^2}\right]
\]

For \(\beta \ll 1\), we obtain Larmor’s differential formula:

\[
\frac{dP}{d\Omega} = \frac{q^2}{16\pi^2\varepsilon_0c} \beta^2 \sin^2 \theta
\]

where \(\theta\) is the angle between \(\hat{\beta}\) and \(\hat{R}\)

Exercises:

1. Show that the angle which maximizes the radiation is given by:

\[
\theta_{max} = \cos^{-1} \left(\frac{1}{3\beta} \left[\sqrt{1+15\beta^2} - 1\right]\right) \approx \frac{1}{2\gamma} = \frac{\sqrt{1-\beta^2}}{2}
\]
ELECTRODYNAMICS: PHYS 30441

2. Show that the total power of a non-relativistic particle is given by the Larmor formula:

$$\bar{P} = \frac{q^2 \beta^2}{6\pi e_o c}$$

In Q1. we isolate the $\theta$ dependence and differentiate w.r.t. to $\theta$:

$$\frac{d}{d\theta} \left[ \frac{\sin^2 \theta}{(1-\beta \cos \theta)^2} \right] = \frac{2 \cos \theta \sin \theta}{(1-\beta \cos \theta)^2} - \frac{5 \beta \sin^3 \theta}{(1-\beta \cos \theta)^3}$$

and set this to zero at a turning point (max, min):

$$\frac{2 \cos \theta \sin \theta}{(1-\beta \cos \theta)^2} - \frac{5 \beta \sin^3 \theta}{(1-\beta \cos \theta)^3} = 0$$

$$\Rightarrow \left[ 2 \cos \theta (1-\beta \cos \theta) - 5 \beta \sin^2 \theta \right] \sin \theta = 0$$

or $$(2x - 2\beta x^2) - 5\beta (1-x^2) = 0$$ (with $x = \cos \theta$)

$$3\beta x^2 + 2x - 5\beta = 0$$

$$x = \frac{-2 \pm \sqrt{4 + 4.3\beta^2 .5}}{2.3\beta} = \frac{-1 \pm \sqrt{1+15\beta^2}}{3\beta}$$

and we find: $\theta_{\text{max}} = \cos^{-1} \left[ \frac{-1+\sqrt{1+15\beta^2}}{3\beta} \right]$ 

In Q2. we integrate over the solid angle $d\Omega$ ($= \sin \theta d\theta d\phi$):

$$\bar{P}_L = \frac{q^2 \beta^2}{16\pi e_o c} \int_{\theta=0}^{\theta=\pi} d\phi \int_{\phi=0}^{\phi=\pi} \sin^2 \theta \sin \theta d\theta = \frac{q^2 \beta^2}{16\pi e_o c} 2\pi \int_0^1 (1-\cos^2 \theta) d\cos \theta = \frac{q^2 \beta^2}{8\pi e_o c} \left( \frac{2-2}{3} \right) = \frac{q^2 \beta^2}{8\pi e_o c} \frac{4}{3}$$

$$\Rightarrow \bar{P}_L = \frac{q^2 \beta^2}{6\pi e_o c}$$
Radiation from a charged particle with acceleration perpendicular to velocity

We start from the general formula for radiation:

\[ \frac{d\overrightarrow{P}}{d\Omega} = \frac{q^2}{16\pi^2 c_e c} \left( \frac{\overrightarrow{R} \times (\overrightarrow{R} - \beta \hat{n}) \times \dot{\beta}}{\left(1 - \beta \cdot \overrightarrow{R}\right)^2} \right) \]

First we calculate the numerator:

\[ \overrightarrow{R} \times \left((\overrightarrow{R} - \beta \hat{n}) \times \dot{\beta}\right) = \overrightarrow{R} \times \left(\overrightarrow{R} \times \dot{\beta}\right) - \overrightarrow{R} \times (\beta \times \dot{\beta}) \]

\[ (\overrightarrow{R} \cdot \dot{\beta})\overrightarrow{R} - (\overrightarrow{R} \cdot \beta)\dot{\beta} + (\overrightarrow{R} \cdot \beta)\dot{\beta} \]

\[ \overrightarrow{R} \cdot \beta (\overrightarrow{R} - \beta) - \beta (1 - \overrightarrow{R} \cdot \beta) \]

\[ \left[\overrightarrow{R} \times \left((\overrightarrow{R} - \beta) \times \dot{\beta}\right)\right]^2 = (\overrightarrow{R} \cdot \dot{\beta})^2 (1 + \beta^2) - 2(\overrightarrow{R} \cdot \beta)^2 (1 - \overrightarrow{R} \cdot \beta) + \beta^2 (1 - \overrightarrow{R} \cdot \beta)^2 \]

(\beta \perp \beta \text{ and } \beta \cdot \dot{\beta} = 0 \text{ has been used})

\[ = (\overrightarrow{R} \cdot \dot{\beta})^2 (1 + \beta^2 - 2\overrightarrow{R} \cdot \beta - 2\overrightarrow{R} \cdot \dot{\beta}) + \beta^2 (1 - \overrightarrow{R} \cdot \beta)^2 \]

\[ \Rightarrow \left[\overrightarrow{R} \times \left((\overrightarrow{R} - \beta) \times \dot{\beta}\right)\right]^2 = \left(\overrightarrow{R} \cdot \dot{\beta}\right)^2 (1 - \beta^2) + \beta^2 (1 - \overrightarrow{R} \cdot \beta)^2 \]

Now, \( \overrightarrow{R} \cdot \dot{\beta} = \beta \cos\chi = \beta \sin\theta \cos\phi \) and \( \overrightarrow{R} \cdot \beta = \beta \cos\theta \)

\[ \Rightarrow \left[\overrightarrow{R} \times \left((\overrightarrow{R} - \beta) \times \dot{\beta}\right)\right]^2 = \left(\beta \sin\theta \cos\phi\right)^2 (1 - \beta^2) + \beta^2 (1 - \beta \cos\theta)^2 \]

Also, the denominator is given by:

\[ (1 - \beta \cdot \overrightarrow{R})^3 = (1 - \beta \cos\theta)^3 \]

\[ \Rightarrow \frac{d\overrightarrow{P}}{d\Omega} = \frac{q^2}{16\pi^2 c_e c} \left[ \frac{(\overrightarrow{R} \times (\overrightarrow{R} - \beta) \times \dot{\beta})^2}{(1 - \beta \cdot \overrightarrow{R})^3} \right] = \frac{q^2}{16\pi^2 c_e c} \frac{-(\beta \sin\theta \cos\phi)^2 (1 - \beta^2) + \beta^2 (1 - \beta \cos\theta)^2}{(1 - \beta \cos\theta)^3} \]

Thus, the radiated power per unit solid angle for \( \beta \perp \dot{\beta} \) (synchrotron radiation) is:

\[ \frac{d\overrightarrow{P}}{d\Omega} = \frac{q^2}{16\pi^2 c_e c (1 - \beta \cos\theta)^3} \left[ \frac{\beta^2}{1 - \sin^2\theta \cos^2\phi} \right] \]

\[ \frac{1}{\gamma^2 (1 - \beta \cos\theta)^3} \]

\[ = \frac{1}{\gamma^2 (1 - \beta \cos\theta)^3} \]
Exercises:

1. Show that there is no synchrotron radiation for $\phi = 0$ and $\theta_{\text{min}} = \cos^{-1}\beta$. Also show that the maximum radiation occurs for $\phi = 0$ and $\theta_{\text{max}} = 0$.

2. Integrate $d\mathbf{P}_||/d\Omega$ and $d\mathbf{P}_\perp/d\Omega$ over $d\Omega$ to obtain the total radiated power:

$$\mathbf{p}_|| = \gamma^2 \mathbf{p}_L, \quad \mathbf{p}_\perp = \gamma^4 \mathbf{p}_L; \quad \mathbf{p}_L = \frac{q^2 \beta^2}{6\pi \epsilon_0 c}$$

In Q1 consider $\phi = 0$ and solve for $\theta$:

$$\frac{1}{1 - \beta \cos \theta} \left[ 1 - \frac{\sin^2 \theta}{\gamma^2 (1 - \beta \cos \theta)^2} \right] = 0$$

or

$$\frac{1}{1 - \beta \cos \theta} \left[ (1 - \beta \cos \theta)^2 - (1 - \cos^2 \theta)(1 - \beta^2) \right] = 0$$

$$\Rightarrow \frac{1}{1 - \beta \cos \theta} \left[ 1 + (\beta \cos \theta)^2 - 2\beta \cos \theta - (1 - \beta^2) \cos \theta (1 - \beta^2) \right] = 0$$

$$\frac{1}{1 - \beta \cos \theta} \left[ \beta^2 - 2\beta \cos \theta + \cos^2 \theta \right] = \frac{1}{(1 - \beta \cos \theta)^3} (\beta - \cos \theta)^2 = 0$$

$$\Rightarrow \theta = \theta_{\text{min}} = \cos^{-1}\beta$$

To prove the power is maximized consider $\phi = 0$, diff wrt to $\theta$ and set equal to zero for the turning points. This can also be seen directly as the last term in parenthesis is maximized when $\theta = 0$.

In Q.2 firstly we integrate over the solid angle $\Omega$.
\[ d\vec{p} = \frac{q^2}{16\pi^2 \varepsilon_0 c} \left[ \frac{\beta^2 \sin^2 \theta}{(1 - \beta \cos \theta)^3} \right] d\Omega = \frac{q^2}{16\pi^2 \varepsilon_0 c} \left[ \frac{\beta^2 \sin^2 \theta}{(1 - \beta \cos \theta)^3} \right] d\Omega d\phi \]

\[ \vec{p}_1 = \frac{q^2}{16\pi^2 \varepsilon_0 c} \int \left[ \frac{\beta^2 \sin^2 \theta}{(1 - \beta \cos \theta)^3} \right] d\Omega d\phi = \frac{q^2}{16\pi^2 \varepsilon_0 c} \beta^2 2\pi \int_0^\pi \sin \theta d\theta \left[ \frac{\sin \theta \cos \phi}{(1 - \beta \cos \theta)^3} \right] = \frac{q^2}{8\pi\varepsilon_0 c} \beta^2 \int_{-1}^{1} \left( 1 - x^2 \right) \left( 1 - \beta x \right) \left( 1 - \beta x \right)^{-\frac{3}{2}} \frac{1}{(1 - \beta x)^3} dx = \frac{q^2}{8\pi\varepsilon_0 c} \beta^2 \frac{4}{3} \left( 1 - \beta^2 \right)^{\frac{3}{2}} \]

\[ \vec{p}_2 = \frac{q^2}{6\pi\varepsilon_0 c} \beta^2 \gamma^6 = \vec{p}_1 \gamma^6 \]

\[ \vec{p}_3 = \frac{q^2}{6\pi\varepsilon_0 c} \beta^2 \gamma^6 = \vec{p}_2 \gamma^6 \]

**Charged particle in circular orbit**

High energy colliders such as LEP2 and LHC at CERN, and the Tevatron at FNAL accelerate particle in a circular orbit. The accelerated particles are constricted to a circular orbit by applying a constant magnetic field (as illustrated in the adjacent Fig.).

The acceleration is transverse to the motion: \( a = \frac{\gamma^2}{R} \) and this makes:

\[ \vec{p}_1 = \frac{q^2}{6\pi\varepsilon_0 c} \beta^2 \gamma^6 = \frac{q^2 a^2}{6\pi\varepsilon_0 c} \gamma^6 = \frac{q^2 c^3 B^2}{6\pi\varepsilon_0} R^{-2} \gamma^6 \]

Equating the centripetal force to the Lorentz force:

- \( \gamma m a = q \vec{v} \times \vec{B} \rightarrow \gamma m \frac{\gamma^2}{R} = q \vec{v} \vec{B} \rightarrow R = \frac{m \gamma^3 \vec{v} \vec{B}}{q} \)

\[ \vec{p}_3 = \frac{q^2 B^2 \beta^2 \gamma^6}{6\pi\varepsilon_0 m^2 c} \quad m = \text{particle's mass and } B = \text{applied magnetic field.} \]

**Properties of Circular Accelerators and Synchrotron Radiation**

- Circular accelerators are efficient in that the particles repeatedly pass the same accelerating sections.
- The maximum guiding magnetic B-field which minimizes R is fixed according to the constraints of magnet technology; e.g. superconducting magnets at LHC
- The synchrotron radiation is more severe for lighter particles. How much is the ratio of the increase in synchrotron radiation?
Exercise

1. Show that the radiation from a particle moving in a circle is larger by a factor of $\gamma^2$ than the radiation from a linearly accelerated particle for the same applied force $F = \frac{dp}{dt}$, with $p = m \gamma \vec{v}$, i.e. show that

   $\bar{P}_\parallel = \frac{q^2}{6\pi\varepsilon_0 c^3} \left| \frac{dp}{dt} \right|^2$ and $\bar{P}_\perp = \frac{q^2 \gamma^2}{6\pi\varepsilon_0 c^3} \left| \frac{dp}{dt} \right|^2$

Ans: First consider collinear acceleration (i.e. linacs), in this case

   $\bar{P}_\parallel = \frac{q^2}{6\pi\varepsilon_0 c} \beta^2 \gamma^6 = \frac{q^2}{6\pi\varepsilon_0 m^2 c^3} m^2 v^2 \gamma^6$, 

and evaluating the corresponding momentum:

   $\frac{dp}{dt} = m \frac{d}{dt} \frac{v}{\sqrt{1-(v/c)^2}} = m \left( \vec{v} \gamma + v \frac{v}{c^2} \right) \left( \frac{1}{1-(v/c)^2} \right)^{1/2} = m \left( \vec{v} \gamma + \beta \gamma^3 \right)$

Thus, $\frac{dp}{dt} = m \gamma v (1+\beta^2 \gamma^2) = m \gamma v \left( 1 + \frac{\beta^2}{1-\beta^2} \right) = m \gamma^2 v$

This makes the energy radiated in a collinear accelerator:

\[
\bar{P}_\parallel = \frac{q^2}{6\pi\varepsilon_0 m^2 c^3} \left( \frac{dp}{dt} \right)^2
\]

In the case of circular acceleration the energy radiated is:

\[
\bar{P}_\perp = \frac{q^2}{6\pi\varepsilon_0 c} \beta^2 \gamma^4 = \frac{q^2}{6\pi\varepsilon_0 m^2 c^3} m^2 v^2 \gamma^4
\]

Again, we evaluate the momentum,

$\frac{dp}{dt} = m \frac{d}{dt} \frac{v}{\sqrt{1-(v/c)^2}} = mv \gamma$, 

where we have used the fact that in circular motion $dv/dt = 0$. Thus the power radiated is given by,

\[
\bar{P}_\perp = \frac{q^2}{6\pi\varepsilon_0 m^2 c^3} m^2 v^2 \gamma^4 = \frac{q^2}{6\pi\varepsilon_0 m^2 c^3} \gamma^2 \left( \frac{dp}{dt} \right)^2
\]

For the same applied force, $F = \frac{dp}{dt}$, the energy radiated is $\gamma^2$ larger in circular motion compared to that in linear motion. Clearly this makes linear accelerators (linacs) a necessity for high energy electron/positron colliders, such as the International Linear Collider (ILC), which aims at a center of mass energy of 500 GeV and for the Compact Linear Collider (CLIC) which aims at a centre of mass energy of 3 TeV.